Systems Innovation 1
– General System Theory –

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What is system?

- Features of system
  - Inhomogeneous (often hierarchical)
  - Interactions between components
  - Global order
  - Goal-oriented (in case of human-made system)
- Examples of system
  - Solar system, organisms, brain
  - Power plant, automobile factory, Internet
  - Ecology of Tokyo bay, global environment
  - Economy of Japan, medical insurance, university

System is the way how you look at object

- What do you see?
  - Young lady
  - Old woman
- What organization is the Univ. of Tokyo?
  It provides/produces …
  - Higher education
  - Flesh human resources
  - Salary
  - Foods and entertainment
  - Lots of waste

General system theory (1)
– Allometric principle –

- Relation observed in various areas of biology

$$y = bx^{\alpha}$$

$$\frac{dy}{dt}\frac{1}{y} = \alpha \frac{dx}{dt}\frac{1}{x}$$

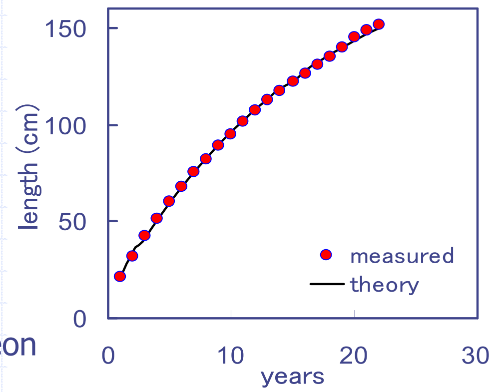
- Constant growth ratio α between x and y
- Familiar also in economy (α : elasticity)

General system theory (2)

– Bertalanffy's growth equation –

- Fits growth curves of almost species

$$\frac{dw}{dt} = \eta w^{\alpha} - \kappa w$$



Growth of sturgeon

State equations

- General description of dynamic system

$$\frac{dQ_1}{dt} = f_1(Q_1, Q_2, \cdots, Q_n)$$

$$\frac{dQ_2}{dt} = f_2(Q_1, Q_2, \cdots, Q_n)$$

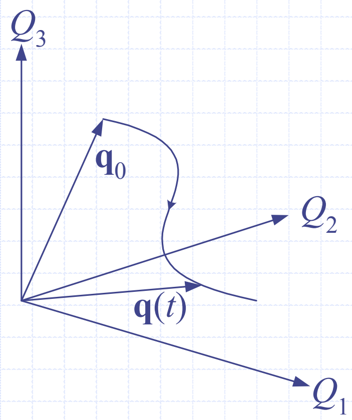
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{dQ_n}{dt} = f_n(Q_1, Q_2, \cdots, Q_n)$$

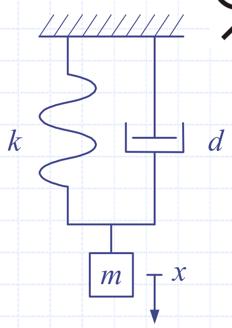
$Q_1, Q_2, \ldots, Q_n$: State variables

State space and trajectory

- State space
  - Phase space composed of state variables
  - Each state is represented as a point in the state space
- Trajectory
  - Shows dynamic behaviour of the system



Example in kinetics



$$m\ddot{x} = -kx - d\dot{x}$$

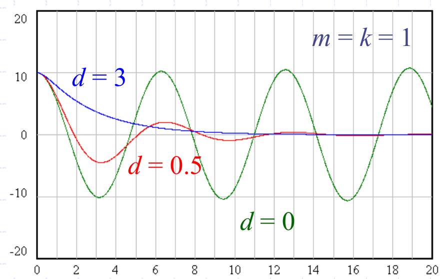
$$\ddot{x} = \frac{1}{m}(-kx - d\dot{x})$$

- State variables

$$Q_1 = x, \quad Q_2 = \dot{x}$$

$$\dot{Q}_1 = Q_2$$

$$\dot{Q}_2 = -\frac{k}{m}Q_1 - \frac{d}{m}Q_2$$

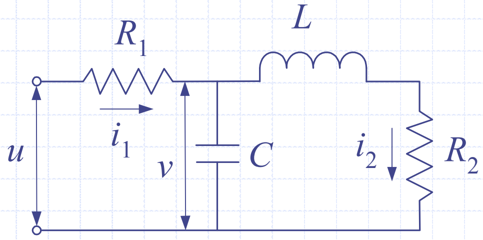


Example in electronics

$$u = R_1 i_1 + v$$

$$v = R_2 i_2 + L\frac{di_2}{dt}$$

$$i_1 = i_2 + C\frac{dv}{dt}$$



$$R_1 CL\frac{d^2 i_2}{dt^2} + (R_1 R_2 C + L)\frac{di_2}{dt} + (R_1 + R_2) i_2 = u$$

$$x_1 = i_2, x_2 = \dot{i}_2$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \frac{-1}{R_1 CL}\begin{bmatrix}0 & 1\\ R_1 + R_2 & R_1 R_2 C + L\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \frac{1}{R_1 CL}\begin{bmatrix}0\\ 1\end{bmatrix}u$$

Example in biochemistry

$$X \xrightarrow{K_1} x_1 \underset{k_2}{\overset{k_1}{\rightleftarrows}} x_2 \xrightarrow{k_3} x_3 \xrightarrow{K_3} \text{discharge}$$

$$\frac{dx_1}{dt} = K_1(X - x_1) - k_1 x_1 + k_2 x_2$$

$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2 - k_3 x_2$$

$$\frac{dx_3}{dt} = k_3 x_2 - K_3 x_3$$

Example in world politics

Richardson model

- Arms race between two opponents
  - Armament in proportion to opponent's action
  - Disarmament due to the load to national economy
  - Latent incentive toward armament

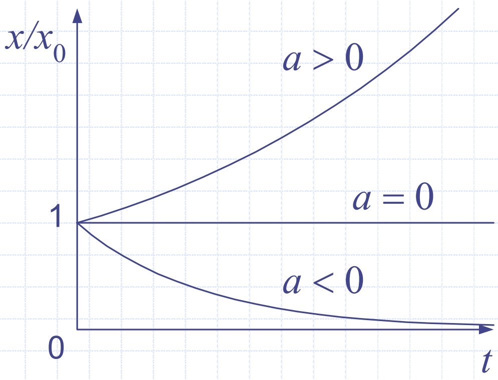
$$\frac{dx}{dt} = ky - \alpha x + g$$

$$\frac{dy}{dt} = lx - \beta y + h$$

First order linear system

$$\frac{dx}{dt} = ax$$

$$x(t) = x_0 e^{at}$$



Second order linear system

- State equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

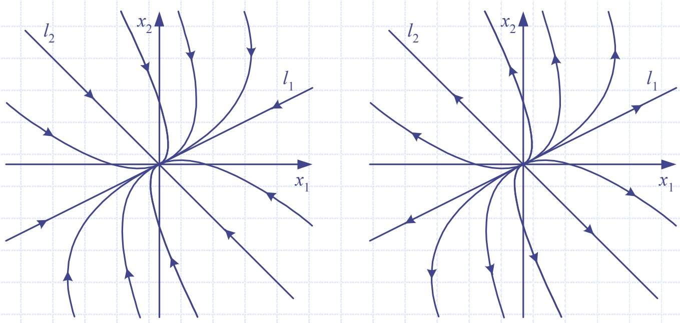
- Eigenvalues

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - a & -b \\ -c & \lambda - d \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

$$D = (a+d)^2 - 4(ad - bc)$$

Two real eigenvalues (D>0)

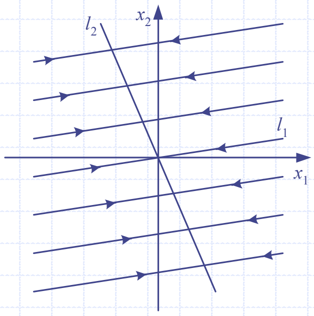
$$x_i(t) = w_{i1}e^{\lambda_1 t} + w_{i2}e^{\lambda_2 t} \qquad (i = 1,2)$$



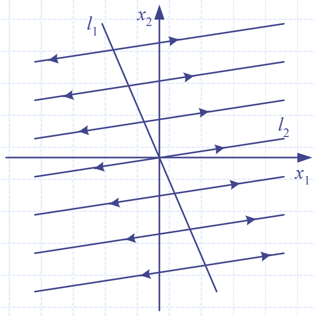
$\lambda_1 < \lambda_2 < 0$ $\qquad$ $0 < \lambda_1 < \lambda_2$

Two real eigenvalues (D>0)

$$x_i(t) = w_{i1}e^{\lambda_1 t} + w_{i2}e^{\lambda_2 t} \qquad (i = 1,2)$$



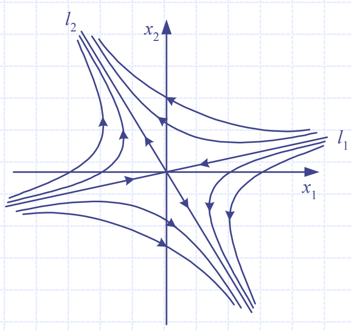
$\lambda_1 < \lambda_2 = 0$



$0 = \lambda_1 < \lambda_2$

Two real eigenvalues (D>0)

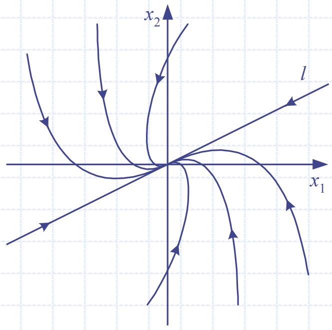
$$x_i(t) = w_{i1}e^{\lambda_1 t} + w_{i2}e^{\lambda_2 t} \qquad (i = 1,2)$$



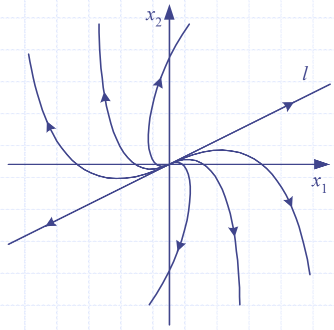
$\lambda_1 < 0 < \lambda_2$

Single real eigenvalue (D=0)

$$x_i(t) = (w_{i1} + w_{i2}t)e^{\lambda t} \qquad (i = 1, 2)$$



$\lambda < 0$

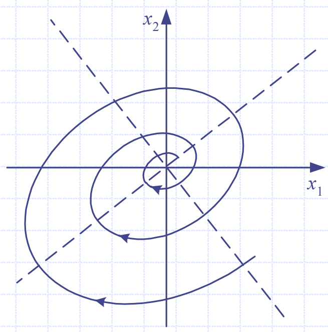


$0 < \lambda$

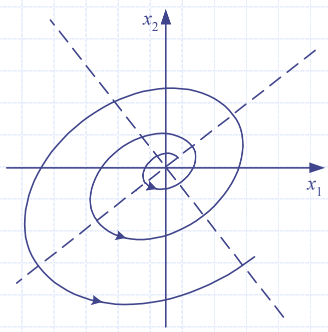
Conjugate complex eigenvalues (D<0)

$$\lambda = \mu_1 \pm \mu_2 j$$

$$x_i(t) = e^{\mu_1 t}(w_{i1} \sin \mu_2 t + w_{i2} \cos \mu_2 t) \qquad (i = 1, 2)$$



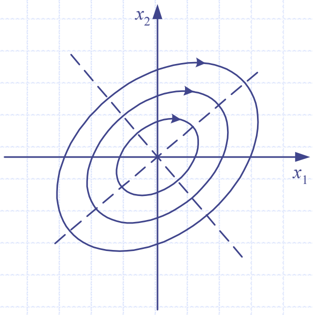
$\mu_1 < 0$



$0 < \mu_1$

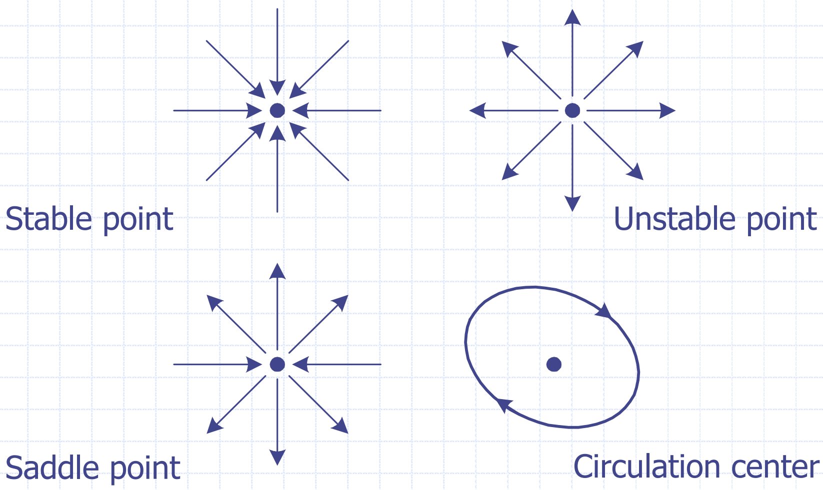
Conjugate complex eigenvalues ($D<0$)

$$x_i(t) = w_{i1} \sin \mu_2 t + w_{i2} \cos \mu_2 t \qquad (i = 1, 2)$$



$$\mu_1 = 0$$

Behaviour around equilibrium point



Kinetic equation (1)

- System description with nth-order ODE

$$\frac{d^n x}{dt^n} + a_1 \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_n x = 0$$

- State variables can be defined as follows

$$\begin{aligned}
Q_1 &= x \\
Q_2 &= \dot{Q}_1 && = \frac{dx}{dt} \\
Q_3 &= \dot{Q}_2 + a_1 \dot{x} && = \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
Q_n &= \dot{Q}_{n-1} + a_{n-2}\dot{x} = \frac{d^{n-1}x}{dt^{n-1}} + a_1 \frac{d^{n-2}x}{dt^{n-2}} + \cdots + a_{n-2}\frac{dx}{dt}
\end{aligned}$$

Kinetic equation (2)

- Rewrite kinetic equation

$$\dot{Q}_n = \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_{n-2}\frac{d^2 x}{dt^2} = -a_{n-1}\frac{dx}{dt} - a_n x$$

- Equivalent state equations

$$\begin{aligned}
\dot{Q}_1 &= Q_2 \\
\dot{Q}_2 &= -a_1 Q_2 + Q_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\dot{Q}_{n-1} &= -a_{n-2} Q_2 - Q_n \\
\dot{Q}_n &= -a_{n-1} Q_2 - a_n Q_1
\end{aligned}$$

Example of population dynamics
– Lotka-Volterra model –

◈ Two species with competition
- Limitation of resources
- Prey-predator relation

$$\frac{dN_1}{dt} = (\varepsilon_1 - \lambda_1 N_1 - \mu_1 N_2) N_1$$
$$\frac{dN_2}{dt} = (\varepsilon_2 - \mu_2 N_1 - \lambda_2 N_2) N_2$$

λ : intraspecific μ : interspecific competition coeff.

Behavior of L-V model (1)

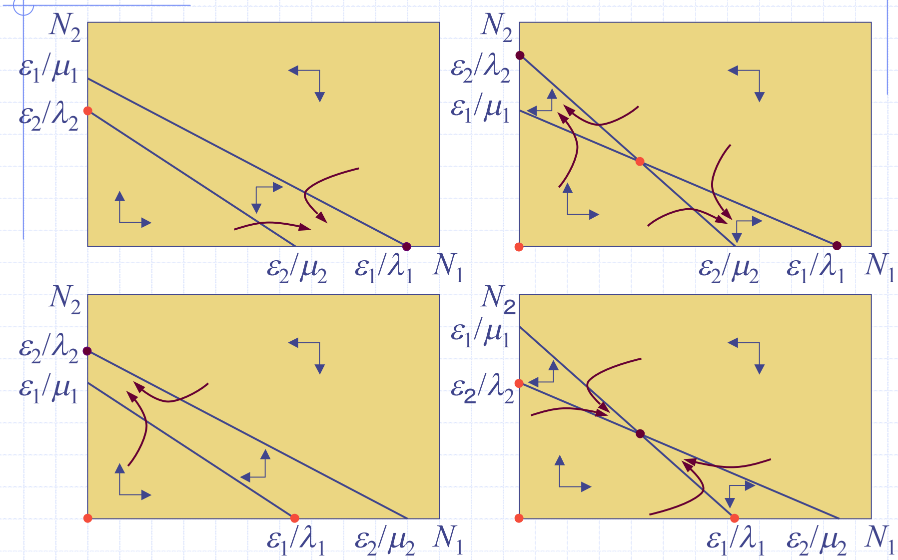
$$dN_1/dt = 0 \quad :$$
$$N_1 = 0 \quad \text{or} \quad \lambda_1 N_1 + \mu_1 N_2 = \varepsilon_1$$
$$dN_2/dt = 0 \quad :$$
$$N_2 = 0 \quad \text{or} \quad \mu_2 N_1 + \lambda_2 N_2 = \varepsilon_2$$

◈ Four fixed (equilibrium) points

$\mathrm{P}_0 : (0, 0)$ $\mathrm{P}_1 : (0, \varepsilon_2/\lambda_2)$

$\mathrm{P}_2 : (\varepsilon_1/\lambda_1, 0)$ $\mathrm{P}_3 : (N_1^*, N_2^*)$

Behavior of L-V model (1)



Prey-predator system

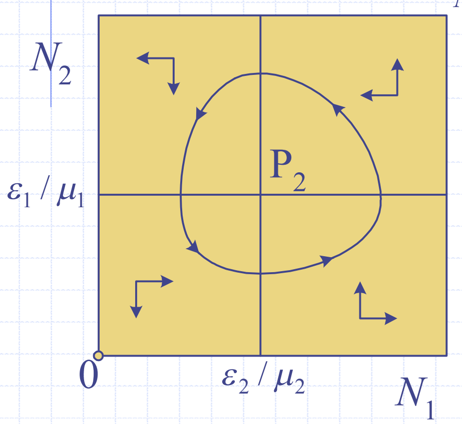
- No intraspecific competition

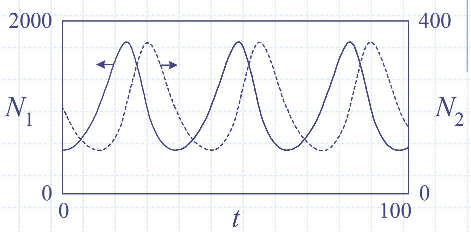
$$\frac{dN_1}{dt} = (\varepsilon_1 - \mu_1 N_2) N_1$$

$$\frac{dN_2}{dt} = (-\varepsilon_2 + \mu_2 N_1) N_2$$

- Prey (N_1) and predator (N_2)
  - Predator hunts and eats prey.
  - Predator cannot survive without prey.
  - Predator breeds rapidly, if prey is affluent.

Behavior of predator-prey system





$$(\varepsilon_1 = 0.2, \varepsilon_2 = 0.2,$$
$$\mu_1 = 0.001, \mu_2 = 0.0002)$$

With intraspecific competition

$$\frac{dN_1}{dt} = (\varepsilon_1 - \lambda_1 N_1 - \mu_1 N_2) N_1$$

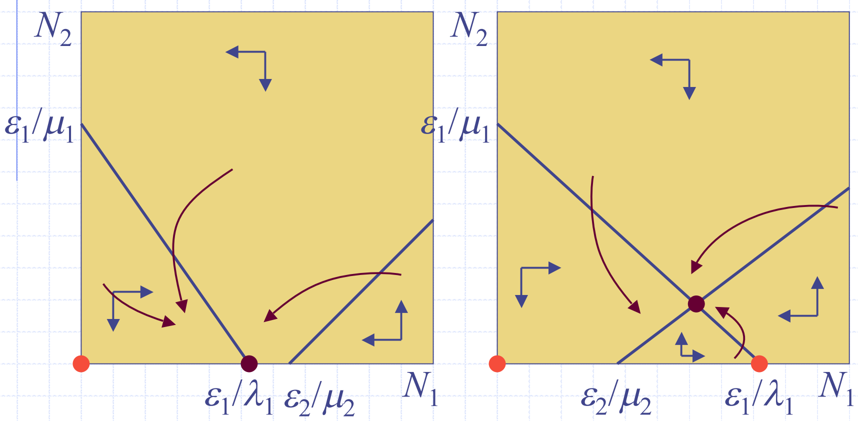
$$\frac{dN_2}{dt} = (-\varepsilon_2 + \mu_2 N_1 - \lambda_2 N_2) N_2$$

- Three fixed points

$$P_0 : (0, 0)$$

$$P_1 : (\varepsilon_1 / \lambda_1, 0) \quad P_2 : (N_1^*, N_2^*)$$

Behavior with intraspecific competition

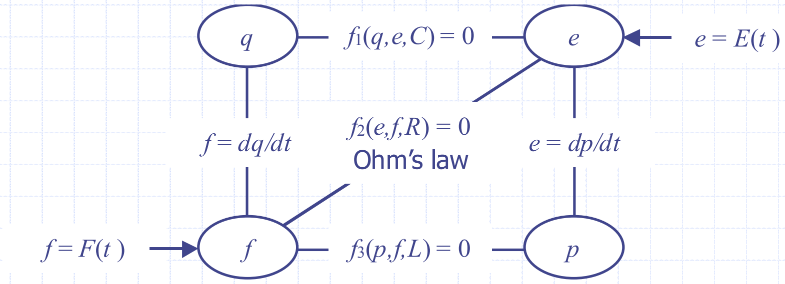


General model of dynamic system

- Isomorphism in different fields of science
  One-to-one correspondence between the elements of different systems
  - Movement in kinetic system (physics)
  - Chemical reaction (chemistry)
  - Growth of organisms (biology)
  - Population in ecosystem (ecology)
  - Growth of capital and economy (economy)
- Existence of a general model that explains behaviour of dynamic systems including non-physical phenomena

General law of physics

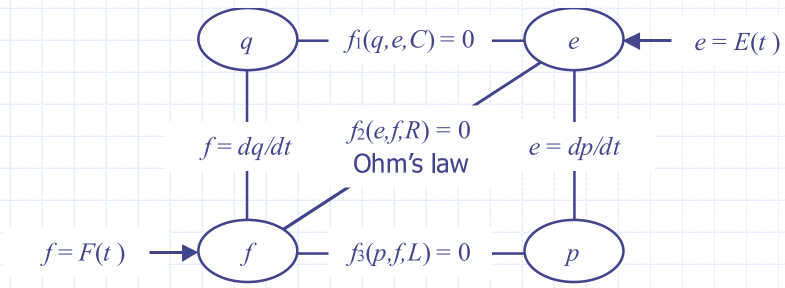
– Tetrahedron of states –



quantity	kinetic	electrical	hydraulic	thermal
effort (e)	force (N)	voltage (V)	pressure (Pa)	temp.diff. (K)
flow (f)	velocity (m/s)	current (A)	flow (m^3/s)	heat flux (W)
momentum (p)	momentum (N·s)	mag.flux (V·s)		
displacement (q)	displacement (m)	charge (C)	volume (m^3)	heat (J)

Extended general law of physics

– General law of flow –



quantity	diffusion	trade	finance	culture
effort (e)	density gradient	abundance of goods	investment potential	creativity
flow (f)	mass flux	goods flow	money	information
displacement (q)	density	stock	stock	soft power

Interaction and reaction rate

- Interaction
  - System components make impacts on each other
    - Collision, gravity, chemical interaction, communication, sexual reproduction, trading, international conflict, ...
  - Direct/remote, bi-directional/unidirectional
- Reaction
  - Interactions result in some changes in system states
- Reaction rate
  - Times or density of reactions that occur per unit time
  - Increases when the number of relevant entities increase

First-order reaction

- A system component changes its status spontaneously, and usually decays into another entity.

  Reaction rate constant k

  Probability that the component decays per unit time

  $$\frac{dX}{dt} = -kX \qquad X = X_0 e^{-kt}$$

  Half-life $T_{1/2}$ Time required for a quantity to fall to half

  $$T_{1/2} = \frac{\ln 2}{k}$$

Example of first-order reaction

- Decay of radioactive nucleus

  $^{60}Co \longrightarrow {}^{60}Ni + e^{-}$ $T_{1/2}$=5.27yr

$$\lambda = \frac{\ln 2}{5.27 \times 365 \times 24 \times 60 \times 60} = 4.17 \times 10^{-9} s^{-1}$$

- Failure of components
  - Reliability at time t with a constant failure rate λ

$$P = e^{-\lambda t}$$

Second-order reaction

- Component X and Y directly interacts to yield new components
  - The reaction rate is proportional to the product of the number (density) of X and that of Y

$$\frac{dX}{dt} = \frac{dY}{dt} = -kXY$$

  - Second-order reaction by two Xs

$$\frac{dX}{dt} = -kX^2$$

$$X = \frac{X_0}{1 + X_0 kt}$$

$$\boxed{\begin{aligned} &-\frac{dX}{X^2} = kdt \\ &-\int_{X_0}^{X} \frac{dX}{X^2} = \int_0^t k\,dt \\ &\frac{1}{X} - \frac{1}{X_0} = kt \end{aligned}}$$

Example of second-order reaction

- Hunting in Lotka-Volterra model
- Purchase of goods triggered by word of mouth
- Chemical reaction

  X + Y → Z
- Sexual reproduction
  - Proportional to the chance of pairing

    $$B = amXY = am\nu(1-\nu)N^2$$

Higher-order reaction

- αX + βY + γZ + ... →
  - Progress of reactions

    $$\xi = \frac{X_0 - X}{\alpha} = \frac{Y_0 - Y}{\beta} = \frac{Z_0 - Z}{\gamma} = \cdots$$
  - Reaction rate

    $$v = \frac{d\xi}{dt} = kX^p Y^q Z^r \cdots$$
  - Total order of reaction

    $$n = p + q + r + \cdots$$

Zero-order reaction

- Reaction rate does not depend on the number or density of relevant entities.

$$\frac{dX}{dt} = -k_0$$

$$X = X_0 - k_0 t$$

Initial density

http://www.cse.sys.t.u-tokyo.ac.jp/furuta/teaching/si/