Resilience Informatics for Innovation

— Classical Decision Theory —

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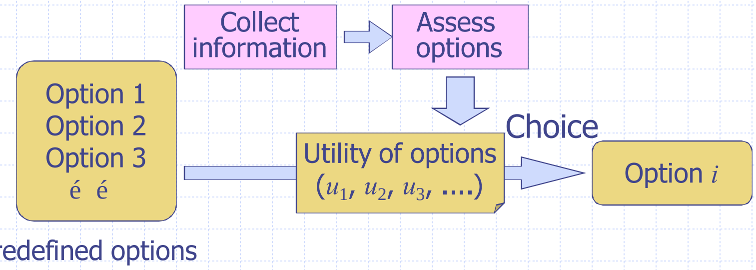
Decision-making and rationality

- What is decision-making?
  - Methodology for making a choice
  - The quality of decision-making determines success or failure of innovation and business.
- Rational decision-making
  - No means exist that certainly lead us to a correct decision due to uncertainties in the world.
  - Anyhow, we need to decide rationally in some sense, with no regret. ⇒ Max. use of information

Classical decision-making

- Normative model of decision-making
  - Choosing one option from many
  - Utility is what one wants to make maximum in the choice.



Choice and preference

- Preference
  - An act to chose one option from many
  - Basic unit of classical decision
- Order of preference in a dyad

$a \succ b$ — a is preferred to b.

$a \approx b$ — a is equivalent to b.

$a \succeq b$ — a is preferred or equivalent to b.

Utility and utility function

- Utility
  - What one wants to make maximum in the choice
  - Subjective value of each option evaluated by the decision-maker
- Utility function $u(x)$
  - Mapping from the set of options to real numbers

    $a \succeq b \quad \Leftrightarrow \quad u(a) \geq u(b)$
  - Utility function is a tool to deal with the utility mathematically

Condition for rational decision

- In order that the preference is mathematically consistent, it must be a weak order relation.
  - Completeness

    $a \succeq b$ or $a \preceq b$ for any pair of a and b
  - Transitivity

    $a \succeq b, b \succeq c \quad \Rightarrow \quad a \succeq c$

Expected utility

- Definition of expected utility

$$E = \sum_{i=1}^{n} u(s_i) p_i$$

s_i : possible state
p_i : occurrence probability of s_i
$u(s_i)$: utility of s_i

- Expected utility hypothesis
  - The utility of option a under uncertainty is represented by its expected utility $E(a)$.

Example of expected utility

	Win.prob.	Prize	Consolation
Lottery 1	0.01	$15,000	none
Lottery 2	0.02	$8,000	none
Lottery 3	0.01	$10,000	$50

$E(L1)$ = 0.01 x $15,000 = $150
$E(L2)$ = 0.02 x $8,000 = $160
$E(L3)$ = 0.01 x $10,000 + 0.99 x $50 = $149.5

Choice under uncertainty

- **Uncertain situation**

  s_i : Possible state

  p_i : Occurrence probability of s_i

  a_j : Option of action

  u_{ij} : Utility when s_i obtains after a_j has been taken

Various decision criteria (1)

- **Expected utility criterion**

$$a^* = a_j \quad \text{s.t.} \quad U(a_j) = \sum_{i=1}^{n} u_{ij} p_i \ \rightarrow \ \max$$

  - If the occurrence probability distribution is unknown, suppose $p_i = 1/n$ (Laplace criterion)

- **Max-min criterion**

$$a^* = a_j \quad \text{s.t.} \quad U(a_j) = \min_i u_{ij} \rightarrow \ \max$$

  - Maximize the utility for the worst case

Various decision criteria (2)

- Hurwitz criterion

  $$a^* = a_j \text{ s.t. } U(a_j) = \alpha \max_i u_{ij} + (1-\alpha) \min_i u_{ij} \rightarrow \max$$

  - $\alpha = 0$ and $\alpha = 1$ correspond to complete pessimism (max-min criterion) and complete optimism.

- Regret criterion (Minimum opportunity loss)

  $$a^* = a_j \quad \text{s.t.} \quad L(a_j) = \max_i r_{ij} \rightarrow \min$$

  $$r_{ij} = \max_j u_{ij} - u_{ij}$$

  - Opportunity loss r_{ij} is the degree of regret compared with the maximum utility obtainable with perfect foresight.

Example of choice under uncertainty (1)

- You are a street food stall owner. Which sells well depends on the weather: ice cream or hot dog. Which will you stock more for tomorrow's business.

Weather	Sunny	Cloudy	Rainy
Probability	0.6	0.3	0.1
Ice cream	1,000	500	200
Half & half	800	700	450
Hot dog	650	600	500

Example of choice under uncertainty (2)

Criterion	Expected	Laplace	Max-min
Ice cream	770	567	200
Half & half	735	650	450
Hot dog	620	583	500

Weather	Hurwitz*	Regret
Ice cream	600	300
Half & half	625	200
Hot dog	575	350

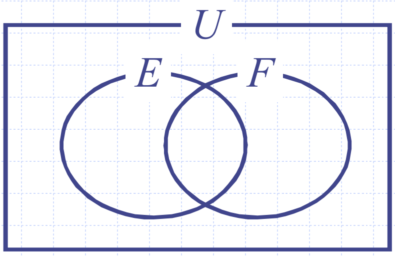
* $\alpha = 0.5$

Conditional probability

U : Sample space

E, F : Subset of U

$|X|$: Size of X



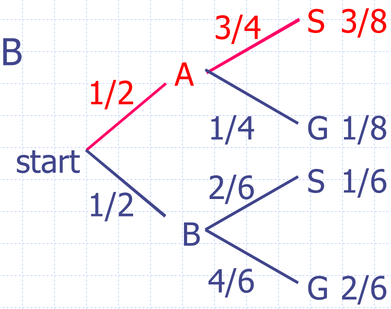
Conditional probability

$$P(E|F) = \frac{|E \cap F|}{|F|} = \frac{|E \cap F||U|}{|F||U|} = \frac{P(E \cap F)}{P(F)}$$

$$P(E \cap F) = P(E)P(F|E) = P(F)P(E|F)$$

Example of conditional probability

- Two invisible bins A and B
  - A : 3 silver & 1 gold coins
  - B : 2 silver & 4 gold coins



- You drew a silver coin from either of the two bins by chance. Which bin did you choose?

$$P(A|S)=\frac{P(A\cap S)}{P(S)}=\frac{3/8}{3/8+1/6}=\frac{9}{13}\approx 0.692$$

Independent events

- When event E and F satisfy the following conditions, they are independent.

$$P(E|F)=P(E) \qquad P(F|E)=P(F)$$

- The following is the necessary and sufficient condition for that E and F are independent.

$$P(E\cap F)=P(E)P(F)$$

Bayes theorem

$$P(F \mid E) = \frac{P(E \mid F)P(F)}{P(E)}$$

- General form of Bayes theorem
  - H_1, H_2, … H_n are exclusive each other and

    $H_1 \cup H_2 \cup \cdots \cup H_n = U$

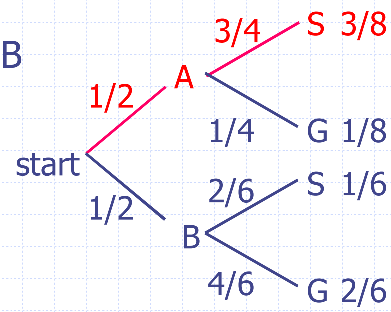
$$P(H_i \mid E) = \frac{P(H_i)P(E \mid H_i)}{P(H_1)P(E \mid H_1) + \cdots + P(H_n)P(E \mid H_n)}$$

Coins-in-bins problem revisited

- Two invisible bins A and B

  A : 3 silver & 1 gold coins

  B : 2 silver & 4 gold coins



  - You drew a silver coin from either of the two bins by chance. Which bin did you choose?

$$P(A \mid S) = \frac{P(S \mid A)P(A)}{P(S \mid A)P(A) + P(S \mid B)P(B)} = \frac{3/4 \cdot 1/2}{3/4 \cdot 1/2 + 1/3 \cdot 1/2} = \frac{9}{13}$$

Bayesian inference (1)

H : Hypothesis

E : Evidence, testimony, or symptom

$P(H)$: Prior probability with no evidence

$P(H|E)$: Posterior probability after evidence E has been obtained

Bayesian inference (2)

- After E has been obtained, how we should modify the probability of H?

$$P(H \mid E) = \frac{P(E \mid H)P(H)}{P(E)} \qquad P(\overline{H} \mid E) = \frac{P(E \mid \overline{H})P(\overline{H})}{P(E)}$$

- Modification of odds in terms of E

$$O(H \mid E) = \frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{P(E \mid H)}{P(E \mid \overline{H})}\frac{P(H)}{P(\overline{H})} = \lambda_E O(H)$$

Posterior odds ($O(H \mid E)$) — Prior odds ($O(H)$)

Bayesian inference (3)

- After another evidence F has been obtained,

$$O(H \mid E \cap F) = \frac{P(H \mid E \cap F)}{P(\overline{H} \mid E \cap F)}$$

$$= \frac{P(E \mid H)}{P(E \mid \overline{H})} \frac{P(F \mid H)}{P(F \mid \overline{H})} \frac{P(H)}{P(\overline{H})} = \underline{\lambda_E \lambda_F} O(H)$$

Modification factor for each evidence

Hasty doctor example

- You received a cancer test. The doctor said that 1 out of 1,000 is the average rate of cancer at your age. The test is accurate and it gives a correct result for 99% of cancer patients and 97% of non cancer patients. Your test result was positive, and the doctor recommended you hospitalization ASAP.

$$O(C|+) = \frac{P(+|C)}{P(+|\overline{C})} \frac{P(C)}{P(\overline{C})} = \frac{0.99 \times 0.001}{0.03 \times 0.999} \approx \frac{1}{30}$$

Three prisoners problem
(Monty Hall problem)

- Out of three prisoners, two will be executed and one will be freed tomorrow. Prisoner A heard from the jailor that B will be executed. A was delighted that his alive probability has increased from 1/3 to 1/2 with this information, since either A or C will be freed.

$$P(A|b) = \frac{P(A)P(b|A)}{P(A)P(b|A) + P(B)P(b|B) + P(C)P(b|C)}$$
$$= \frac{1/3 \cdot 1/2}{1/3 \cdot 1/2 + 1/3 \cdot 0 + 1/3 \cdot 1} = \frac{1}{3}$$

Component failure example

- Failure statistics of a particular component are collected for every 100 days of operation, and the following data were obtained. Evaluate the failure rate.

Interval	1	2	3	4
Times of failures	6	3	5	6

Average = (6+3+5+6)/4/100 = 0.05 day^{-1}
STD = 0.02 day^{-1}

Probability density function

- Cumulative distribution function $F(x)$
  - Probability that a random variable X takes a value not greater than x

$$F(x) = P(-\infty < X \le x)$$

- Probability density function $f(x)$

$$f(x) = \frac{dF(x)}{dx} \qquad F(x) = \int_{-\infty}^{x} f(x)dx$$

$$f(x)dx \approx P(x < X \le x + dx)$$

Bayesian inference on distribution

$f(\theta)$: Prior density function of θ

$f(\theta | A)$: Posterior density function of θ

- Bayesian update of the density function after event A has been observed

$$f(\theta | A) = \frac{P(A|\theta) f(\theta)}{\int P(A|\theta) f(\theta)\, d\theta}$$

Application of Bayesian approach to component failure example

- Poisson distribution

$$P(k \mid \lambda) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

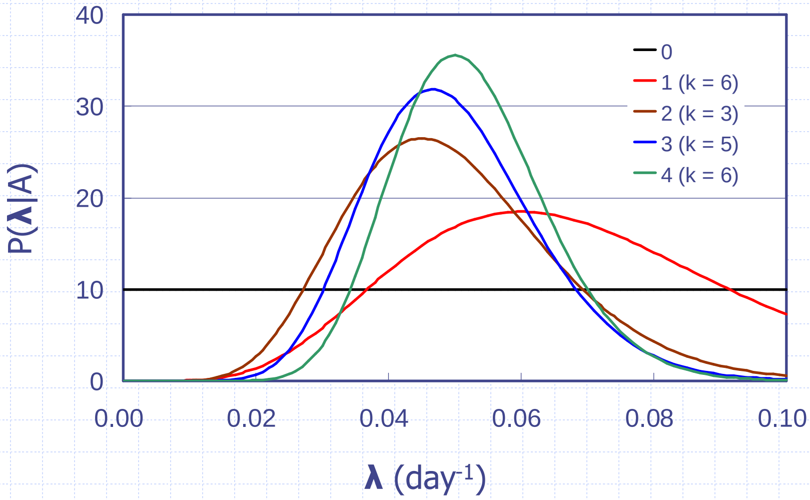
λ : Failure rate

T : Interval of observation (100 days)

k : Times of failures observed

- Uniform distribution in [0, 0.1] is assumed for $f(\lambda)$ at the beginning.

Bayesian update of failure rate



Concepts of probability (1)

- Normative definition (Pascal)
  - Fraction of number of cases among the sample space

$$p = |A| / |U|$$

- Statistical definition (d'Alembert)
  - Asymptotic value of event frequency

$$p = \lim_{n \to \infty} (f_n / n)$$

Concepts of probability (2)

- Subjective definition (Bayes)
  - Degree of individual confidence on occurrence of the event

- Informatics definition (Shannon)
  - Amount of information that implies occurrence of the event

$$S = -\log_2 p$$